Lemma. *$\mathcal{M}_F$ is closed under $\mu^*(X \ominus X)$.*

Operations are "continuous" with respect to this "metric". We say $A_n \to A$ in outer measure means that $\mu^*(A_n \ominus A) \to 0$ as $n \to \infty$

Proposition. *If $A_n \to A$, $B_n \to B$ then $A_n \cup B_n$, $A_n \cap B_n$, $A_n \backslash B_n$, A_n^c converge to where you would think they do*

Proof. Long and tedious, its in the book.

Proposition. *If $A, B \in 2^X$ and $\mu^*(A) < \infty$ then*

$$|\mu^*(A) - \mu^*(B)| \leq \mu^*(A \ominus B) = d(A, B)$$

□

Proof. Assume that $\mu^*(A), \mu^*(A) < \infty$, all of the other cases are easy. And assume without loss of generality that $\mu^*(A) \geq \mu^*(B)$, then

$$\mu^*(A) - \mu^*(B) \leq \mu^*(A \ominus B) \Rightarrow \mu^*(A) \leq d(A, B) + \mu^*(B)$$

and this is true since $A \subset (A \ominus B) \cup B$.

Before we proceed, a useful fact □

$$\boxed{(A \setminus B) = (A \cap B^c)}$$

Definition. The sets of μ-measurable sets are defined as

$$\mathcal{M} = \left\{ A \subset X | A = \bigcup_{i=1}^{\infty} B_i,\ B_i \in \mathcal{M}_F \right\}$$

We would like to show that μ^* is a measure on $\mathcal{M}_F$. We need to prove that $\mathcal{M}_F$ is a ring and that μ^* is countably additive.

Lemma. *$\mathcal{M}_F$ is a ring, i.e. $A, B \in \mathcal{M}_F$, then $A \cup B, A \ominus B \in \mathcal{M}_F$.*

Proof. $A, B \in \mathcal{M}_F$, by assumption there exists approximating sequences $A_i, B_i \in \mathcal{R}$ such that $\mu^*(A_i \ominus A) \to 0$ and $\mu^*(B_i \ominus B) \to 0$.

We compute $\mu^*((A \cup B) \ominus (A_i \cup B_i)) = (A \cup B) \setminus (A_i \cup B_i) \cup (A_i \cup B_i) \setminus (A \cap B)$. We just compute one side of this union,because the argument would be symmetric, and we already know that the union will be in the ring.

$$\begin{aligned}(A \cup B) \setminus (A_i \cup B_i) &= (A \cup B) \cap (A_i \cup B_i)^c = (A \cup B) \cap (A_i^c \cap B_i^c) \\ &= (A \cap A_i^c \cap B_i^c) \cup (B \cap A_i^c \cap B_i^c) \subset (A \setminus A_i) \cup (B \setminus B_i)\end{aligned}$$

and also $(A_i \cup B_i) \setminus (A \cup B) \subset (A_i \setminus A) \cup (B_i \setminus B)$. So the entire set contains $(A \ominus A_i) \cup (B \ominus B_i)$. So then

$$\mu^*(A \cup B, A_i \cup B_i) \leq \mu^*(A \ominus A_i, B \ominus B_i) \to 0$$

so $A \cup B \in \mathcal{M}_F$. Also like to show that $A \ominus B \in \mathcal{M}_F$. Since $(A \ominus B) = (A \setminus B) \cup (B \setminus A)$. It suffices to show that $A \setminus B \in \mathcal{M}_F$, since we know that the union is in $\mathcal{M}_F$ already. We again compute half of $\mu^*((A \setminus B) \ominus (A_i \setminus B_i))$, that is $((A \setminus B) \setminus (A_i \setminus B_i))$

$$\begin{aligned}(A \setminus B) \setminus (A_i \setminus B_i) &= (A \cap B^c) \cap (A_i \cap B_i^c)^c = (A \cap B^c) \cap (A_i^c \cup B_i) \\ &= ((A \cap B^c \cap A_i^c) \cup (A \cap B^c \cap B_i)) \subseteq (A \setminus A_i) \cup (B_i \setminus B) \subseteq A \ominus A_i \cup B \ominus B_i\end{aligned}$$

if we work out the other have we get the same thing, so

$$((A \setminus B) \setminus (B \setminus B_i)) \subseteq (A \ominus A_i) \cup (B \ominus B_i)$$

and so

$$\mu^*((A \setminus B) \setminus (B \setminus B_i)) \subseteq \mu^*(A \ominus A_i) + \mu^*(B \ominus B_i) \to 0$$

□

In order that μ^* is a measure, it must be finite, so we want to show that $\mu^* < \infty$ for all $A \in \mathcal{M}_F$.

Lemma. $\mu^* < \infty$

Proof.

$$|\mu^*(A) - \mu^*(A_i)| \leq \mu^*(A \ominus A_i) = d(A, A_i) \to 0, \qquad \mu^*(A_i) = \mu(A_i)$$

So $\mu^*(A) \leq \mu(A_i) + 1$ for some i □

Lemma. $\mu^*(A \cup B) + \mu^*(A \cap B) = \mu^*(A) + \mu^*(B)$

Proof. We know this for $A_i, B_i \in \mathcal{R}$ that is

$$\mu^*(A_i \cup B_i) \to \mu^*(A \cup B), \qquad \mu^*(A_i \cap B_i) \to \mu^*(A \cap B)$$

and we know that

$$|\mu^*(A \cup B) - \mu^*(A_i \cup B_i)| \leq \mu^*((A \cup B) \ominus (A_i \cup B_i))$$

So now we have a finitely additive measure. □